

LOCAL STABILITY OF HYDROSTATIC COMPRESSION STATES OF NON-LINEARLY THERMO-VISCO-ELASTIC BODIES OF DIFFERENTIAL TYPE*

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The local Lyapunov stability of a hydrostatically stressed state of an isotropic, homogeneous, non-linearly thermo-visco-elastic body of differential type with complexity n and constant initial temperature is investigated in a dynamical formulation. The connection between the thermal and mechanical fields is taken into account. As a special case a model of an incompressible non-linearly visco-elastic body is investigated. Conditions that ensure the existence and uniqueness of generalized solutions of the linearized equations of motion and heat conduction and their vanishing with time are formulated. The example shows that violation of the conditions obtained can lead to the exponential growth of the solution.

In the investigation of the local stability of hydrostatically stressed states (HSSs) of non-linearly elastic bodies by static methods /1, 2/ the absence of adjacent equilibrium forms came to the fore as a necessary requirement placing restrictions on the form of the equation of state of the material. The stability of the solutions of the equilibrium equations of non-linear visco-elastic bodies with integral defining relations has been investigated /3-5/. The conditions for the stability of an HSS of a visco-elastic body were obtained in /4/. The influence of the temperature field was taken into account in /5/. On the basis of the theory developed in /6/, the behaviour with time of solutions of the equations of motion of linear visco-elastic bodies of differential type has been investigated /7/.

1. The fundamental relations describing a homogeneous isotropic non-linearly thermo-visco-elastic body of differential type with complexity n have the form /8/

$$\begin{aligned} \psi &= \psi(G, \Theta), \quad \eta = -\partial\psi/\partial\Theta(G, \Theta), \quad T = T_0(\Theta, F) + T_1(\Theta, F, A_1, \dots, A_n) \\ T_0 &= 2\rho C^T \cdot \partial\psi/\partial G \cdot C, \quad h = h(\Theta, \nabla\Theta, F), \quad G = C \cdot C^T, \quad F = C^T \cdot C \end{aligned} \quad (1.1)$$

Here ψ and η are the mass densities of the free energy and entropy, T is the Cauchy stress tensor, T_0 and T_1 are the equilibrium and dissipative stress tensors, h is the heat flux vector, Θ is the temperature, C is the deformation gradient, G and F are the Cauchy-Green and Finger deformation measures, ρ is the density of the body in the current configuration, ∇ is the gradient operator in the current configuration, and the A_k ($k = 1, \dots, n$) are Rivlin-Ericksen tensors, defined by the formula /9/

$$A_k = C^{-1} \cdot (d^k G / dt^k) \cdot C^{-T}, \quad k = 1, \dots, n \quad (1.2)$$

Instead of the tensors A_k one can use other indifferent tensors characterizing the deformation velocities in relations (1.1). The tensor T_1 vanishes for $A_k = 0$ ($k = 1, \dots, n$).

When considering a purely mechanical theory one uses a model of an incompressible visco-elastic material with defining relations for the Cauchy stress tensor of the form

$$T = -pE + T_0(F) + T_1(F, A_1, \dots, A_n) \quad (1.3)$$

where p is the unknown function determined from the equations of motion and the incompressibility condition $\det G = 1$, and E is the unit tensor.

An equilibrium state of the body which is in an HSS and has a uniform temperature field will be called a (p, Θ°) -configuration. A (p, Θ°) -configuration of an isotropic body satisfies the relations $T = -pE$, $\Theta = \Theta^\circ$, $C = lQ$, $G = F = l^2E$, $A_k = 0$ and $h = 0$, where Q is a proper orthogonal tensor and $0 < l \leq 1$. For an incompressible material $l = 1$.

In order to investigate the stability of this state, the equations of motion and heat conduction are linearized about the (p, Θ°) -configuration. In the metric of the deformed state the linearized equations have the form /1, 9/

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$$\begin{aligned} \nabla \cdot \Pi &= \rho \partial_t^2 \mathbf{w}, \quad \Pi = \delta \mathbf{T} + \mathbf{T} \operatorname{tr} \mathbf{L} - \mathbf{L}^T \cdot \mathbf{T}, \quad \mathbf{L} = \nabla \mathbf{w} \\ \delta \mathbf{T} &= \frac{d}{d\xi} \mathbf{T}(\mathbf{R} + \xi \mathbf{w}, \Theta^\circ + \xi \theta) \Big|_{\xi=0}, \quad \partial_t = \frac{\partial}{\partial t} \end{aligned} \quad (1.4)$$

Here \mathbf{R} is the radius-vector in the (p, Θ°) -configuration, \mathbf{w} is a small additive displacement vector field and θ is the temperature shift. The symbol δ denotes linear perturbations caused by the displacements and temperature shifts.

Linearizing the Rivlin-Ericksen tensors in the neighbourhood of the (p, Θ°) configuration we obtain

$$\delta \mathbf{A}_k = 2 \partial_t^k \boldsymbol{\varepsilon}, \quad 2 \boldsymbol{\varepsilon} = \mathbf{L} + \mathbf{L}^T \quad (1.5)$$

Following /2/, we transform the equations of motion linearized in the neighbourhood of the (p, Θ°) -configuration to the form

$$\nabla \cdot \mathbf{S} = \rho \partial_t^2 \mathbf{w}, \quad \mathbf{S} = \delta \mathbf{T} - \boldsymbol{\Omega} \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\Omega}, \quad 2 \boldsymbol{\Omega} = \mathbf{L}^T - \mathbf{L} \quad (1.6)$$

Taking account of relations (1.1) and (1.5) and the isotropy conditions, one can show that \mathbf{S} is a linear isotropic function of the tensors $\partial_t^k \boldsymbol{\varepsilon}$ ($k = 0, \dots, n$) and θ :

$$\mathbf{S} = \Lambda(\partial_t) \operatorname{tr} \boldsymbol{\varepsilon} \mathbf{E} + 2M(\partial_t) \boldsymbol{\varepsilon} - \alpha(l) \theta \mathbf{E} \quad (1.7)$$

$$\Lambda(\xi) = \sum_{k=0}^n \lambda_k(l) s^k, \quad M(s) = \sum_{k=0}^n \mu_k(l) s^k$$

Relations (1.7) define the response of the material to small deformations near the HSS. For an incompressible body

$$\mathbf{S} = -\delta p \mathbf{E} + 2M(\partial_t) \boldsymbol{\varepsilon} \quad (1.8)$$

where δp is an unknown function of the coordinates, for which the linearized incompressibility condition $\nabla \cdot \mathbf{w} = 0$ serves as a supplementary defining condition.

To obtain linearized heat-conduction equations we consider the energy-balance equation in the case when there are no bulk heat sources /8-12/:

$$\rho d(\psi + \Theta \eta) / dt = -\nabla \cdot \mathbf{h} + \operatorname{tr}(\mathbf{T} \cdot \nabla d\mathbf{R} / dt)$$

Using relation (1.1), this equation can be transformed to the form

$$\rho \Theta d\eta / dt = -\nabla \cdot \mathbf{h} + \operatorname{tr}(\mathbf{T}_1 \cdot \nabla d\mathbf{R} / dt)$$

The last term on the right-hand side of this equation describes the specific dissipation of energy /9, 12/ for non-linear thermo-visco-elastic bodies of differential type. The presence of the dissipative term distinguishes the energy-balance equation under consideration from the heat-conduction equation for non-linear thermo-elastic bodies. Taking account of relation (1.1) and the material isotropy condition, we write down expressions for the entropy and heat flux vector linearized in the neighbourhood of the (p, Θ°) -configuration:

$$\rho \delta \eta = c(l) \theta + \alpha(l) \nabla \cdot \mathbf{w}, \quad \delta \mathbf{h} = -\kappa(l) \theta \nabla$$

In a small neighbourhood of the (p, Θ°) -configuration the dissipative term in the heat-conduction equation, which vanishes in that configuration (because in equilibrium configurations, of which the (p, Θ°) -configuration is a special case, \mathbf{T}_1 and $d\mathbf{R}/dt$ are both zero), has second order of smallness compared with the other terms and vanishes on linearization. The last linearization is of the heat-conduction equation in a neighbourhood of the (p, Θ°) -configuration, and has the form

$$c(l) \Theta^\circ \partial_t \theta = \kappa(l) \nabla \cdot \nabla \theta - \alpha(l) \Theta^\circ \partial_t \nabla \cdot \mathbf{w} \quad (1.9)$$

The derivation of the non-linear heat-conduction equation and its linearization about an unstressed state of visco-elastic bodies of differential type 1 was performed in /12/. The form of the linearized heat-conduction equation obtained in /12/ is similar to (1.9).

The quantities $c(l)$ and $\kappa(l)$ will be regarded as positive. The initial temperature Θ° will without loss of generality be taken to be equal to unity.

The linearized boundary conditions on the surface of the body $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 = \Sigma_4 \cup \Sigma_5$ have the form /2/

$$\begin{aligned} \mathbf{w} \Big|_{\Sigma_1} &= 0, \quad \mathbf{N} \cdot \mathbf{S} \Big|_{\Sigma_1} = 0, \quad \mathbf{N} \cdot \mathbf{w} \Big|_{\Sigma_2} = 0 \\ \mathbf{N} \cdot \mathbf{S} \cdot (\mathbf{E} - \mathbf{N}\mathbf{N}) \Big|_{\Sigma_3} &= 0, \quad \theta \Big|_{\Sigma_3} = 0, \quad \mathbf{N} \cdot \nabla \theta \Big|_{\Sigma_4} = 0 \end{aligned} \quad (1.10)$$

(where \mathbf{N} is the normal to the surface of the body). The Σ_1 component of the surface of the body is completely fixed, a uniform hydrostatic pressure is assigned to the component Σ_2 , on Σ_3 the body is in contact with a smooth solid surface, the temperature is specified on Σ_4 and the heat flux on Σ_5 .

The initial conditions for w and θ are written as follows:

$$\partial_t^k w|_{t=0} = w_k, \quad k = 0, \dots, N \equiv \max(1, n-1), \quad \theta|_{t=0} = \theta_0 \quad (1.14)$$

A solution of the resulting initial-boundary value problems will be sought in the form

$$w = u + \chi(t) \sum_{k=0}^N \frac{t^k}{k!} w_k, \quad \theta = \tau + \chi(t) \theta_0$$

where the infinitely differentiable function $\chi(t) = 1$ in the neighbourhood of zero and $\chi(t) = 0$ for $t > 1$. The functions u and τ satisfy homogeneous initial and boundary conditions and inhomogeneous equations of motion and heat conduction with some fictitious "body forces" f and "heat sources" q .

2. To investigate the resulting initial-boundary value problems we will use the Laplace transformation (with s as the transform parameter).

As in /7/ we introduce the following spaces.

Suppose H is some separable Hilbert space.

$E_k(\gamma, H)$ (where k is an integer and $\gamma \geq 0$) is the space of functions with values in H , analytic in the half-plane $\operatorname{Re} s > \gamma$ and with finite norm

$$\|\varphi\|_{E_k}^2 = \sup_{\sigma > \gamma} \int_{-\infty}^{+\infty} \|\varphi(\sigma + i\tau)\|_{H^2}^2 (1 + |\sigma + i\tau|^{2k}) d\tau$$

$P_k(\gamma, H)$ is a complex space of functions $g(t)$ with values in H , possessing generalized derivatives /10/ on $[0, +\infty)$ up to order k inclusive, such that $\partial_t^m g = 0$ for $t = 0$ ($0 \leq m \leq k$), and a finite norm

$$\|g\|_{P_k}^2 = \int_0^{+\infty} e^{-2\gamma t} \sum_{m=0}^k \|\partial_t^m g\|_{H^2}^2 dt$$

A theorem has been proved /7/ to the effect that the Laplace transformation operator continuously maps the space $P_k(\gamma, H)$ (where $k \geq 0$ is an integer and $\gamma > 0$) onto the space $E_k(\gamma, H)$, and that it is continuously invertible and that its inverse serves as the inverse Laplace transform operator.

Below we shall need the following assertions.

Lemma 2.1. Suppose the roots of the polynomial $P(s) = p_0 + p_1 s + \dots + p_n s^n$ lie in the left half-plane of the complex plane and $p_n > 0$, then for $\operatorname{Re} s \geq 0$ we have the inequality $|P(s)| \geq dp_n (1 + |s|^{2n})^{1/2}$, where $d \leq 1$.

The proof follows from expanding the polynomial $P(s)$ in simple factors and the inequalities $|s - s_k| \geq d(1 + |s|^2)^{1/2}$ with $d < 1$ for $\operatorname{Re} s_k < 0$.

3. We will consider the first initial-boundary value problem for a compressible thermo-visco-elastic body (1.6), (1.7), (1.9), (1.10) (with $\Sigma = \Sigma_1 = \Sigma_4$), which for brevity we shall call problem A.

Let H_1 be the space of complex-valued functions formed by the closure of the set of continuously differentiable functions equal to zero on the boundary ω , in the norm

$$\|\tau\|_{H_1}^2 = \iint \nabla \tau \cdot \nabla \tau d\omega$$

Here and below we shall not specify the domain of integration of ω . H_2 is the space of vector functions each of whose components belong to H_1 . It is obvious that $H_1 = W_2^{0(1)}$ and $H_2 = W_2^{0(1)} W_2^{0(1)} \times W_2^{0(1)}$.

Applying a Laplace transformation to problem A one obtains a boundary-value problem with a parameter - problem B. We shall call $v(s) \in H_2$, $\tau(s) \in H_1$ a generalized solution of problem B, if for arbitrary functions $g \in H_2$ and $\varphi \in H_1$ we have the equalities

$$\iint \{M(s) \nabla v \cdot \nabla \bar{g}^T + [M(s) + \Lambda(s)] \nabla v \cdot \nabla \bar{g} + \rho s^2 v \cdot \bar{g} - \alpha \tau \nabla \cdot \bar{g} - F \cdot \bar{g}\} d\omega = 0 \quad (3.1)$$

$$\iint \{k \nabla \tau \cdot \nabla \bar{\varphi} + c s \tau \bar{\varphi} - \alpha s v \cdot \nabla \bar{\varphi} - Q \bar{\varphi}\} d\omega = 0 \quad (3.2)$$

($F = Lf \in E_0(0, H_2^{-1})$, $Q = Lq \in E_0(0, H_1^{-1})$)

We shall perform the proof of the existence of a generalized solution to problem B in two stages. Regarding the vector v as given, we will investigate the problem of finding the

temperature, taking as the definition of the solution the satisfaction of relation (3.2) by any function $\varphi \in H_1$. Using the Riesz theorem, Eq.(3.2) can be represented in the form of an operator equation in the space H_1 :

$$\begin{aligned} A(s)\tau &\equiv \kappa\tau + csK\tau = \Phi, \quad \Phi = KQ + \alpha sK_1v \\ (KQ, \varphi)_{H_1} &= \iiint Q\bar{\varphi}d\omega, \quad (K_1v, \varphi)_{H_1} = \iiint v \cdot \nabla\bar{\varphi}d\omega \end{aligned} \quad (3.3)$$

The operator K acts continuously from H_1^{-1} into H_1 and the operator K_1 acts continuously from L_2 into H_1 . One can show that $\|K_1\| = 1$.

Lemma 3.1. Suppose $v \in E_n(0, L_2)$. Then Eq.(3.3) is uniquely soluble for $\text{Re } s > 0$ and its solution can be represented in the form

$$\tau = \tau_1 + \tau_2, \quad \tau_1 \in E_0(0, H_1), \quad \tau_2 \in E_{n-1}(0, H_1) \cap E_n(0, L_2)$$

Proof. We have the following equalities:

$$\begin{aligned} (A(s)\tau, \tau)_{H_1} &= P_0(s)(\tau, \tau)_{H_1} = P_1(s)(\tau, \tau)_{L_2} \\ (P_0(s) = \kappa + cks, P_1(s) &= cs + k^{-1}\kappa, k = \|\tau\|_{L_2}^2 \|\tau\|_{H_1}^{-2} \end{aligned}$$

The parameter $k \in (0, k^0)$, where the quantity k^0 depends on the domain ω . Using Lemma 2.1, the inequality

$$\begin{aligned} |(A(s)\tau, \tau)_{H_1}| &\geq \zeta(s) \|\tau\|_{L_2}^2, \quad |(A(s)\tau, \tau)_{H_1}| \geq \kappa \|\tau\|_{H_1}^2 \\ \zeta(s) &= \min(c, \kappa/k^0) (1 + |s|^2)^{1/4} \end{aligned} \quad (3.4)$$

follows from the relations obtained when $\text{Re } s \geq 0$.

It follows from (3.4) that Eq.(3.3) is properly solvable and the operator $A(s)$ has a continuous inverse $A^{-1}(s)$. It is clear that the operator conjugate to $A(s)$ is $A(\bar{s})$. One can show that the equation with the conjugate operator is properly solvable. From this it follows /14/ that Eq.(3.3) is solvable everywhere for $\text{Re } s > 0$. The operator $A(s)$ is an entire operator-valued function. The analyticity of $\tau(s)$ in the domain of analyticity of Φ for $\text{Re } s > 0$ follows from the continuity of A^{-1} .

We denote by τ_1 and τ_2 the solutions of Eq.(3.3), respectively, corresponding to the first and second terms in the expression for Φ . The concluding estimates of the proof follow from inequalities (3.4):

$$\begin{aligned} \|\tau_1\|_{H_1} &\leq \kappa^{-1} \|KQ\|_{H_1}, \quad \|\tau_2\|_{H_1} \leq \kappa^{-1} |\alpha s| \|v\|_{L_2} \\ \|\tau_2\|_{L_2} &\leq \zeta^{-1}(s) |\alpha s| \|v\|_{L_2} \end{aligned}$$

Using the results of Lemma 3.1, Eq.(3.1) can be presented in the form of an operator equation in the space H_2 :

$$\begin{aligned} B(s)v &\equiv B_0(s)v - \alpha^2 s K_4(s)v = K_3F + F' \\ B_0(s)v &\equiv M(s)v + [M(s) + \Lambda(s)]K_2v + \rho s^2 K_3v \\ (K_2v, g)_{H_1} &= \iiint \nabla v \cdot \nabla \bar{g} d\omega, \quad (K_3v, g)_{H_1} = \iiint v \cdot \bar{g} d\omega \\ (K_4v, g)_{H_1} &= \frac{1}{\alpha s} \iiint \tau_2 \nabla \cdot \bar{g} d\omega, \quad (F', g)_{H_1} = \alpha \iiint \tau_1 \nabla \cdot \bar{g} d\omega \end{aligned} \quad (3.5)$$

The operator K_2 is continuous and positive, and takes H_2 into H_2 . It is known /15/ that $\|K_2\| = 1$. The operator K_3 takes H_2^{-1} continuously into H_2 . The operator $K_4(s)$ takes H_2 continuously into H_2 and is completely continuous. The complete continuity of K_4 is a consequence of the inequality

$$\|K_4(s)v\|_{H_1} \leq d_1 (1 + |s|^2)^{-1/4} \|v\|_{L_2}$$

which is obtained from the estimate for τ_2 in the space L_2 which was pointed out in Lemma 3.1 together with Sobolev's imbedding theorem. Here and below the d_k are arbitrary positive constants.

$B_0(s)$ satisfies the equality

$$\begin{aligned} (B_0(s)v, v)_{H_1} &= P(s)(v, v)_{H_1}, \quad P(s) = M(s) + k_2[\Lambda(s) + M(s)] + \\ &\quad \rho k_3 s^2, \quad k_2 = (K_2v, v)_{H_1} \|v\|_{H_1}^{-2} \in (0, 1), \quad k_3 = (K_3v, v)_{H_1} \|v\|_{H_1}^{-2} \in (0, k_3^0) \end{aligned}$$

The stability requirement on the polynomial $P(s)$ for all k_2, k_3 in their domains of

variation was put forward by /7/ as a sufficient condition for the solubility of Eq.(3.5) with $\alpha = 0$. The solubility of Eq.(3.5) with $\alpha \neq 0$ is given by the following lemma.

Lemma 3.2. Suppose the polynomial $P(s)$ is stable, i.e. its roots have strictly negative real parts. Then a number $\alpha^* > 0$ exists such that for $|\alpha| < \alpha^*$ Eq.(3.5) is uniquely soluble for $\text{Re } s \geq 0$ and the solution $v(s) \in E_n(0, H_2)$.

The proof is similar to the proof of Lemma 3.1 and is based on an a priori estimate.

$B(s)$ satisfies the equality

$$(B(s)v, v)_{H_2} = P(s)(v, v)_{H_2} - \alpha^2 s (K_4(s)v, v)_{H_2}$$

from which for $\text{Re } s \geq 0$ there follows a necessary estimate for sufficiently small α :

$$|(B(s)v, v)_{H_2}| \geq \left[d_2(1 + |s|^{2n})^{1/2} - d_3 \frac{\alpha^2 |s|}{(1 + |s|^2)^{1/2}} \right] \|v\|_{H_2}^2$$

It is clear that the inequality obtained is satisfied for sufficiently small or large $|s|$ independently of α .

In the case $n = 1, 2$ an a priori estimate can be obtained independently of α . To do this the vector g in Eq.(3.1) is set equal to sv , in (3.2) $\varphi = \tau$, and relation (3.1) is added to the complex conjugate of (3.2), leading to the equality

$$(\bar{s}B_0(s)v, v)_{H_2} + (A(s)\tau, \tau)_{H_1} = (\bar{s}K_3F, v)_{H_2} + (\overline{KQ}, \tau)_{H_1}$$

which, using previously obtained inequalities with $\text{Re } s \geq 0$, leads to a necessary estimate on v and τ independently of α .

The solution estimates can be strengthened if one makes additional smoothness assumptions about the initial conditions. In particular, suppose w_k and θ_0 are such that $f \in L_2(\omega)$ and $g \in L_2(\omega)$, then, using the results of Lemmas 3.1 and 3.2, and properties of the operators $A(s)$ and $B(s)$, one can show that

$$v \in E_m(0, L_2), \tau \in E_1(0, L_2), m = \max(2, n)$$

The results obtained prove the existence and uniqueness of the generalized solution of problem B. Application of the theorem from /7/ cited in Sect.2 now guarantees the existence and uniqueness of a solution of problem A in the following sense: for every pair of infinitely differentiable functions $g(t) \in H_2$ and $\varphi(t) \in H_1$ which vanish when t is larger than a certain number depending on g and φ , the solution $u(t), \tau(t)$ satisfies the equation

$$\int_0^\infty \iiint \{ \nabla u \cdot M(-\partial_t) \nabla_k^T + \nabla \cdot u [M(-\partial_t) + \Lambda(-\partial_t)] \nabla \cdot g + \rho u \cdot \partial_t^2 g - \alpha \tau \nabla \cdot g + \kappa \nabla \tau \cdot \nabla \varphi - c \tau \partial_t \varphi - \alpha u \cdot \nabla \partial_t \varphi - f \cdot g - g \varphi \} d\omega dt$$

We have thus proved the following theorem.

Theorem 3.1. Suppose $w_k \in H_2, \theta_0 \in H_1$, and the polynomial $P(s)$ is stable. Then there exists $\alpha^* > 0$ (for $n = 1, 2, \alpha^* = \infty$) such that for $|\alpha| < \alpha^*$ problem A has a unique generalized solution $u \in P_{n-1}(0, H_2)$ and $\tau \in P_0(0, H_1)$. If w_k, θ_0 are such that $f \in L_2$ and $g \in L_2$, then $u \in P_m(0, L_2)$ and $\tau \in P_1(0, L_2)$ where $m = \max(2, n)$.

A sufficient condition for the local stability of a (p, θ^0) -configuration follows directly from Theorem 3.1.

Theorem 3.2. Suppose that the conditions of Theorem (3.1) are satisfied. Then for $|\alpha| < \alpha^*$ the solution of the initial-boundary value problem (1.6), (1.7), (1.9)-(1.11) is asymptotically stable in the following sense:

$$\int_M^\infty \{ \|\partial_t^k w\|_{H_2}^2 + \|\partial_t^i w\|_{L_2} + \|\theta\|_{H_1}^2 + \|\partial_t^j \theta\|_{L_2} \} dt \rightarrow 0, \quad M \rightarrow \infty$$

where $k = 0, \dots, n; i = 0, \dots, m; j = 0, 1$; and

$$\|\partial_t^{k-1} w\|_{H_2} \rightarrow 0, \|\partial_t^{i-1} w\|_{L_2} \rightarrow 0, \|\theta\|_{L_2} \rightarrow 0, \quad t \rightarrow \infty$$

uniformly in t .

The last assertion of Theorem 3.2 follows from the inequality /13/

$$d_4 \|\partial_t^j w\|_{H_2}^2 \leq \sum_{k=0}^n \|\partial_t^k w\|_{H_2}^2 dt, \quad 0 \leq j < n$$

For certain conditions on the set of initial data one can make more precise the manner in which $\theta(t)$ tends to zero. In particular, if w_k, θ_0 are such that $q = 0$, (for example, $\nabla \cdot w_k = 0$ and $\theta_0 = 0$), then with the conditions of Theorem 3.1 one can show that $\theta \in P_{m-1}(0, H_1) \cap P_m(0, L_2)$.

4. All the results presented above can be directly carried over to other initial-boundary value problems for compressible thermo-visco-elastic and incompressible visco-elastic bodies. Here we will merely note the necessary changes in the formulations of the theorems of Sect.3.

In the case of a mixed boundary-value problem for a compressible thermo-visco-elastic body the space H_2 is replaced by H_3 , where H_3 is the closure of vector functions continuously differentiable in ω , satisfying homogeneous kinematic boundary conditions ($w|_{\Sigma_1} = 0, N \cdot w|_{\Sigma_2} = 0$), such that w satisfies Korn's inequality /16/ in the norm induced by the scalar product

$$(w, u)_{H_3} = \int \int \int \varepsilon(w) \cdot \varepsilon(u) d\omega$$

The polynomial $P(s)$ is replaced by $P'(s) = 2M(s) + \Lambda(s)k_2 + \rho k_3 s^2$. The parameter $k_2 \in (0, 3]$ as a consequence of the inequality $\|\nabla \cdot w\|_{L_\infty} \leq \sqrt{3} \|w\|_{H_3}$.

If the boundary conditions allow rigid-body displacements of the body, the assertions of Theorems 3.1 and 3.2 with the above changes hold only for the "deformation" part of the displacement orthogonal in H_3 to the rigid displacement vectors.

In the same way one can investigate the case of more general thermal boundary conditions.

The initial-boundary value problem for an incompressible visco-elastic body is treated similarly. Here the spaces H_2 and H_3 are obtained by the closure in corresponding norms of solenoidal vector functions, i.e. those vector functions satisfying the condition $\nabla \cdot w = 0$. For the first boundary-value problem $P(s) = M(s) + \rho k_3 s^2$, and for the others $P'(s) = 2M(s) + \rho k_3 s^2$.

Analysis of the properties of the polynomials $P(s)$ and $P'(s)$ shows that a necessary, and for $n = 1, 2$ also a sufficient, condition for the asymptotic stability of a (p, θ^0) -configuration is that the inequalities $\mu_k > 0, \lambda_k + 2\mu_k > 0 (k = 0, \dots, n)$ for the first boundary-value problem and $\mu_k > 0, 3\lambda_k + 2\mu_k > 0$ for all boundary-value problems with $l \in (0, 1]$. In the case of an incompressible body these boundary conditions are the inequalities $\mu_k > 0$. These inequalities impose definite restrictions on relations (1.7) and (1.8) defining the response of the material to small deformations near the HSS.

The restrictions imposed by the conditions of Theorem 3.1 for $n > 2$ on α are, in general, necessary.

As an example we will consider the problem of the stability of a rectangle with sides a, b , in contact over its entire surface with a smooth rigid surface and with Neumann conditions on the temperature ($\Sigma = \Sigma_3 = \Sigma_4$). A solution $w = u_1 + v_1, \theta$ satisfying the boundary conditions is sought in the form

$$u(x, y, t) = \sum_{n, m=0}^{\infty} u_{nm} e^{\nu_{nm} t} \sin \frac{\pi n}{a} x \cos \frac{\pi m}{b} y, \quad v(x, y, t) = \sum_{n, m=0}^{\infty} v_{nm} e^{\nu_{nm} t} \cos \frac{\pi n}{a} x \sin \frac{\pi m}{b} y, \quad \theta(x, y, t) = \sum_{n, m=0}^{\infty} \theta_{nm} e^{\nu_{nm} t} \cos \frac{\pi n}{a} x \cos \frac{\pi m}{b} y$$

Substituting this into Eqs.(1.6), (1.9) we obtain a linear homogeneous system of algebraic equations in u_{nm}, v_{nm} and θ_{nm} . The condition for its non-trivial solubility can be shown to be the vanishing of the expression

$$D(\nu, \beta, \alpha) = [\rho \nu^3 + \beta M(\nu)] \{ \alpha^2 \beta \nu + (c\nu + \kappa \beta) [\rho \nu^2 + \beta (\Lambda(\nu) + 2M(\nu))] \} \\ \nu = \nu_{nm}, \quad \beta = (\pi n/a)^2 + (\pi m/b)^2$$

If the equation $D(\nu, \beta, \alpha) = 0$ has a root ν_k with positive real part, this means that the problem has an exponentially increasing solution and therefore the (p, θ^0) -configuration is Lyapunov-unstable for the rectangle. It is clear that for $\alpha = 0$ the stability requirement for all β is equivalent to the stability requirement of the polynomial $P(\nu)$ in the conditions of Theorem 3.1.

One can show that for $n = 1, 2$ a necessary and sufficient condition for the stability of $D(\nu, \beta, \alpha)$ for all β and α is the satisfaction of the inequalities $\lambda_k + 2\mu_k > 0, \mu_k > 0 (k = 0, \dots, 2)$. Analysis of the stability of the polynomial $D(\nu, \beta, \alpha)$ for $n = 3$ shows that for some $\alpha^*(\beta)$ the Hurwitz stability criteria for $D(\nu, \beta, \alpha)$ are violated.

Via the parameter β the critical parameter α^* also depends on the dimensions of the

domain. One can show that $\alpha^* = O(\beta)$ for $\beta \rightarrow \infty$ and $\alpha^* = O(\beta^{-1/4})$ for $\beta \rightarrow 0$. Unlike the conditions on λ_k and μ_k , the restrictions on α for $n > 2$ cannot be considered as a restriction on the defining relations for thermo-visco-elastic bodies in the case of small deformations (1.7) because the constant α^* also depends on the domain occupied by the body.

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